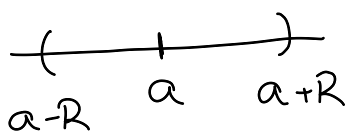
$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

- converge só em $x = a$;
- converge para $x \in \mathbb{R}$;
- converge para $|x-a| < R$ e diverge para $|x-a| > R$.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad , \quad f: D \subset \mathbb{R} \to \mathbb{R}$$

- $D = \{a\}$;
- $D = \mathbb{R}$;
- $(a-R, a+R) \subset D$

  $a-R$, $a+R$ podem ou não fazer parte de D.



Observe que:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

$$\Rightarrow f^{(0)}(a) = c_0 = 0!\, c_0$$

$$f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$\Rightarrow f'(a) = c_1 = 1!\, c_1$$

$$f''(x) = 0 + 2c_2 + 3\cdot 2\cdot c_3(x-a) + 4\cdot 3\cdot c_4(x-a)^2 + \cdots$$

$$\Rightarrow f''(a) = 2c_2 = 2!\, c_2$$

$$f'''(x) = 0 + 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2\cdot c_4(x-a) + \cdots$$

$$\Rightarrow f'''(a) = 3\cdot 2\cdot c_3 = 3!\, c_3$$

$$f^{(4)}(x) = 0 + 4\cdot 3\cdot 2\cdot c_4 + \cdots$$

$$\Rightarrow f^{(4)}(a) = 4\cdot 3\cdot 2\cdot c_4 = 4!\, c_4$$

$$\vdots$$

$$f^{(n)}(a) = n!\, c_n$$

$$\Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

$$[c_1\underbrace{(x-a)}]' = c_1(x-a)' = c_1(1-0)$$

$$[c_2\underbrace{(x-a)^2}]' = c_2[(x-a)^2]' = c_2\cdot 2(x-a)\cdot \cancel{(x-a)'}^{\,1}$$

$$[c_3(x-a)^3]' = c_3\cdot 3(x-a)^2\cdot \cancel{(x-a)'}^{\,1}$$

Série de Taylor centrada em a:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Exemplo: $f(x) = e^x$, $f(1) = ?$

$$f(x) = e^x \Rightarrow f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$f''(x) = e^x \Rightarrow f''(0) = 1$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

$$f(1) = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

$$\approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} = 2 + \frac{1}{2} + \frac{1}{6} = \frac{12+3+1}{6}$$

$$= \frac{16}{6} = \frac{8}{3} \approx 2,666$$

$$\begin{array}{l|l} 8 & 3 \\ \hline 20 & 2,666\dots \\ \ \ 20 & \end{array}$$

Quando $\underline{a=0}$, chamamos a série de potências de série de Maclaurin.

Verificando a convergência:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$$

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{1}{n+1} \cdot x\right| = \frac{1}{n+1} \cdot |x| \xrightarrow{n \to \infty} 0 < 1$$

$\therefore$ a série converge independente do valor de x.

$$D = \mathbb{R}$$

2) $f(x) = \ln x$, $a = 1$

$$f(x) = \ln x \Rightarrow f(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$$

$$f^{(4)}(x) = \frac{-3 \cdot 2}{x^4} \Rightarrow f^{(4)}(1) = -3 \cdot 2$$

$$f^{(5)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5} \Rightarrow f^{(5)}(1) = 4 \cdot 3 \cdot 2$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

$$f^{(n)}(1) = (-1)^{n+1} \cdot (n-1)!$$

$$C_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n!}$$

$$= \frac{(-1)^{n+1}}{n}$$

$$\therefore \ln x = 0 + \frac{1}{1}(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

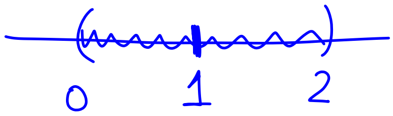
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \cdot (x-1)^n$$

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (x-1)^{n}$$

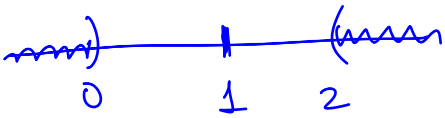
$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}(x-1)^{n}} \right| = \left| (-1) \frac{n}{n+1} (x-1) \right|$$

$$= \frac{n}{n+1} |x-1| = \frac{1}{1+\frac{1}{n}} |x-1| \xrightarrow{n\to\infty} |x-1|$$

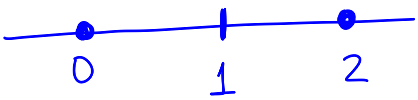
- $|x-1| < 1 \Rightarrow x \in (0,2)$



- $|x-1| > 1$



- $|x-1| = 1$



$x=0$: $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (0-1)^{n} = \sum_{n=1}^{\infty} (-1)^{n+1+n} \cdot \frac{1}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{2n+1} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} -\frac{1}{n}$$ diverge!

$x=2$: $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (2-1)^{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$ (harm. alternada)

converge!